QCD at very high density

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http://theory.fi.infn.it/casalbuoni/barcellona.pdf

http://theory.fi.infn.it/casalbuoni/loff_rev.pdf

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Summary

- Introduction and basics in Superconductivity
- Effective theory
- Color Superconductivity: CFL and 2SC phases
- Effective theories and perturbative calculations
- LOFF phase

Introduction

- Important to explore the entire QCD phase diagram: Understanding of

Hadrons → QCD-vacuum

↓

Understanding of its modifications

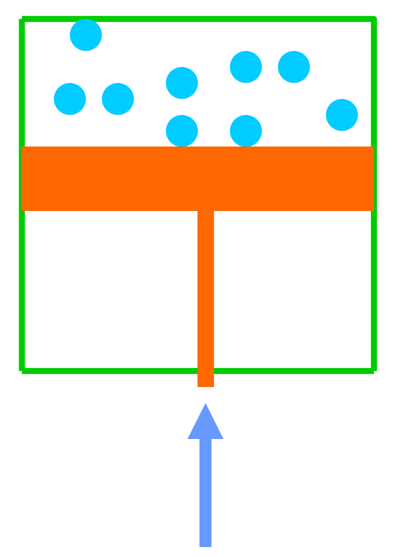
- Extreme Conditions in the Universe:

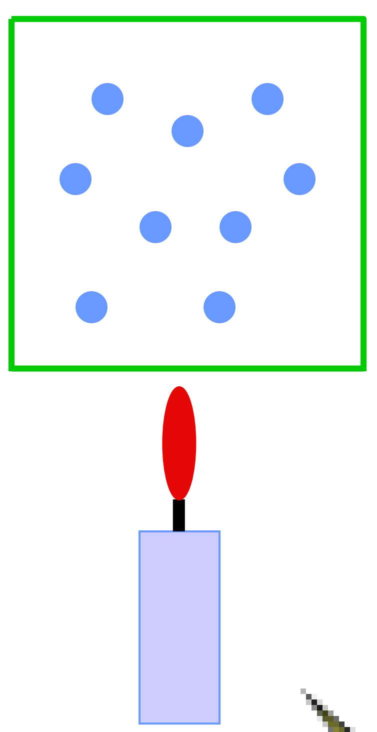
  Neutron Stars, Big Bang

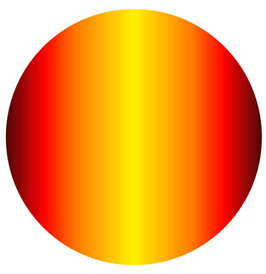
- QCD simplifies in extreme conditions:

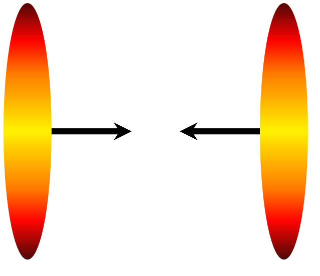
  Study QCD when quarks and gluons are the relevant degrees of freedom

Studying the QCD vacuum under different and extreme conditions may help our understanding





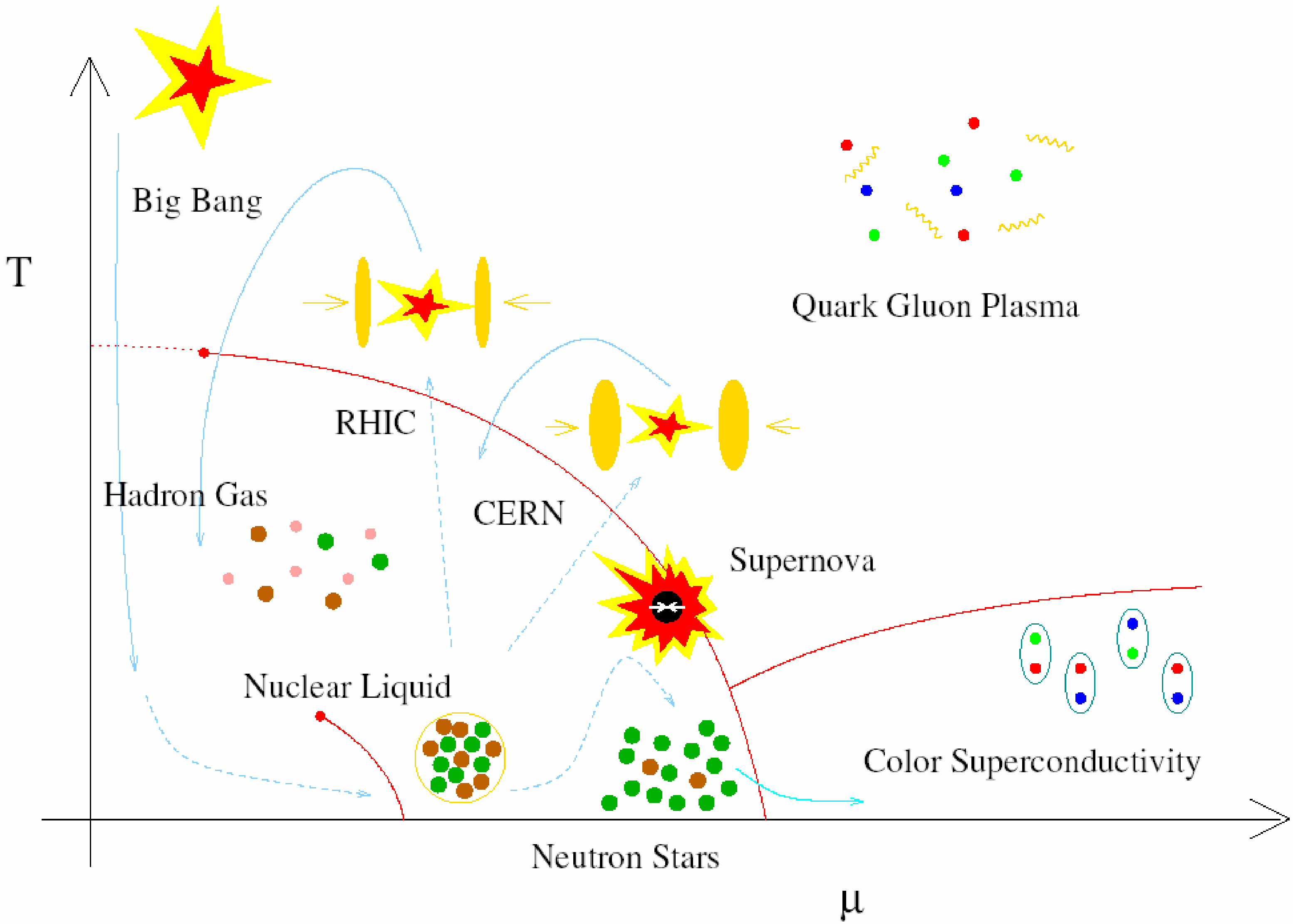




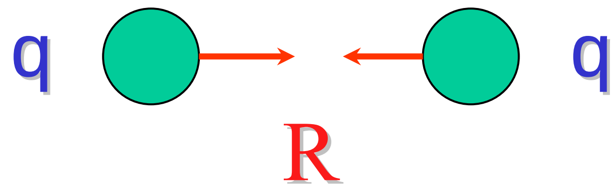
Neutron star

Heavy ion collision

Big Bang


Big Bang
T
Quark Gluon Plasma
RHIC
Hadron Gas
CERN
Supernova
Nuclear Liquid
Color Superconductivity
Neutron Stars
μ

Limiting case $\rho \to \infty \quad (R \to 0)$

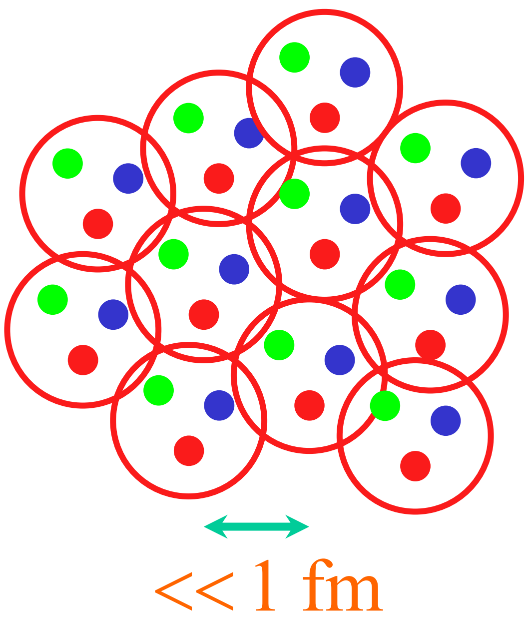


Free quarks

Asymptotic freedom:

When $n_B \gg 1\ fm^{-3}$

free quarks expected

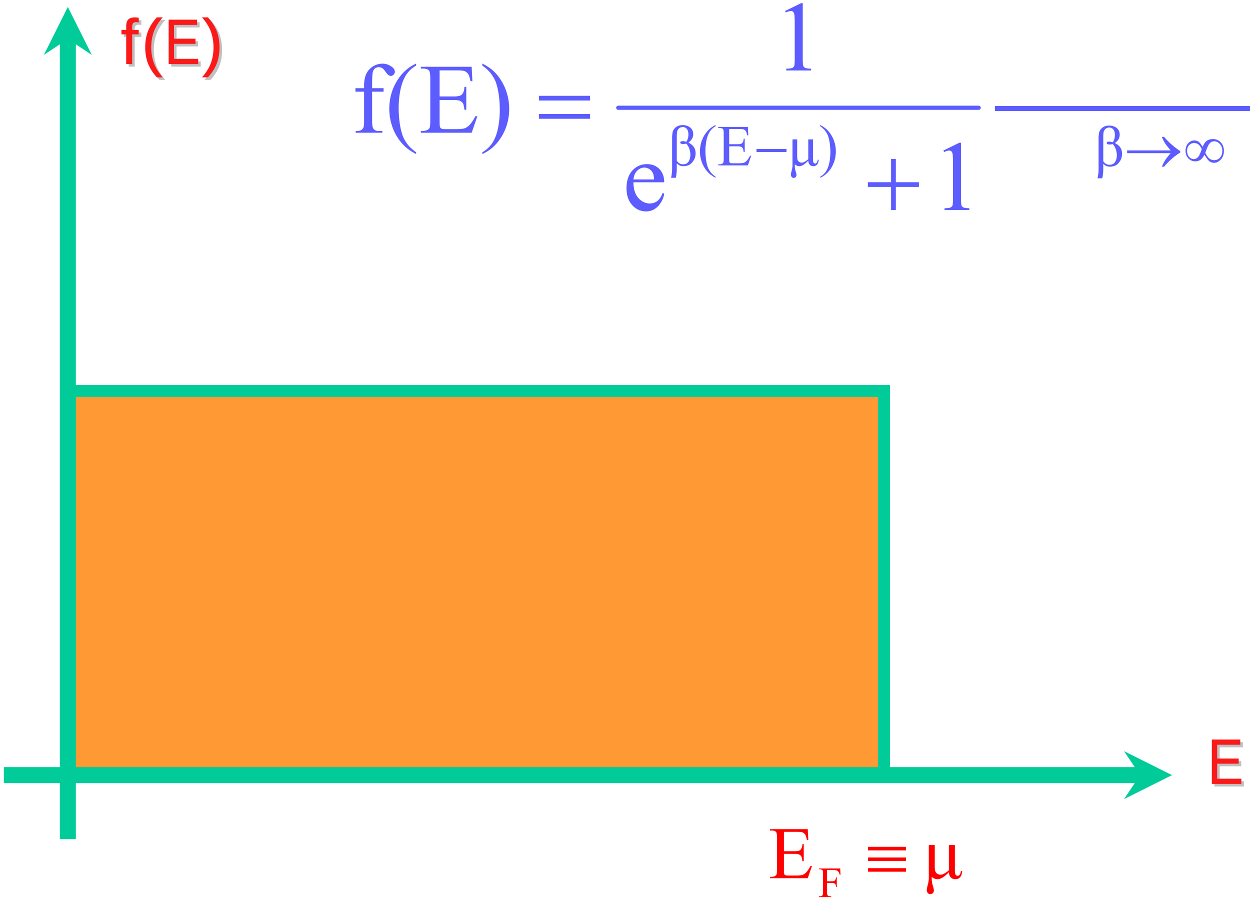


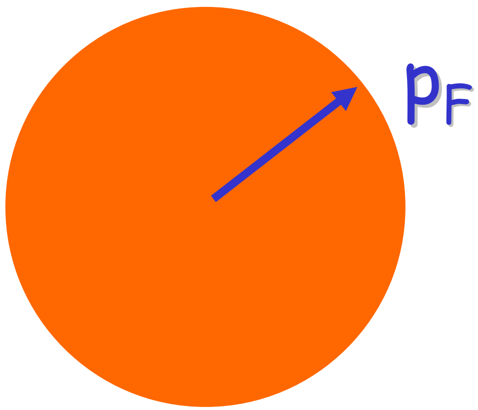
Free Fermi gas and BCS

(high-density QCD)

For $T \to 0 \quad (\beta = 1/kT \to \infty)$

$$f(E) = \frac{1}{e^{\beta(E-\mu)} + 1} \xrightarrow[\beta\to\infty]{} \theta(\mu - E)$$





- High density means high p_F
- Typical scattering at momenta of order of p_F

For $p_F >> \Lambda_{QCD}$

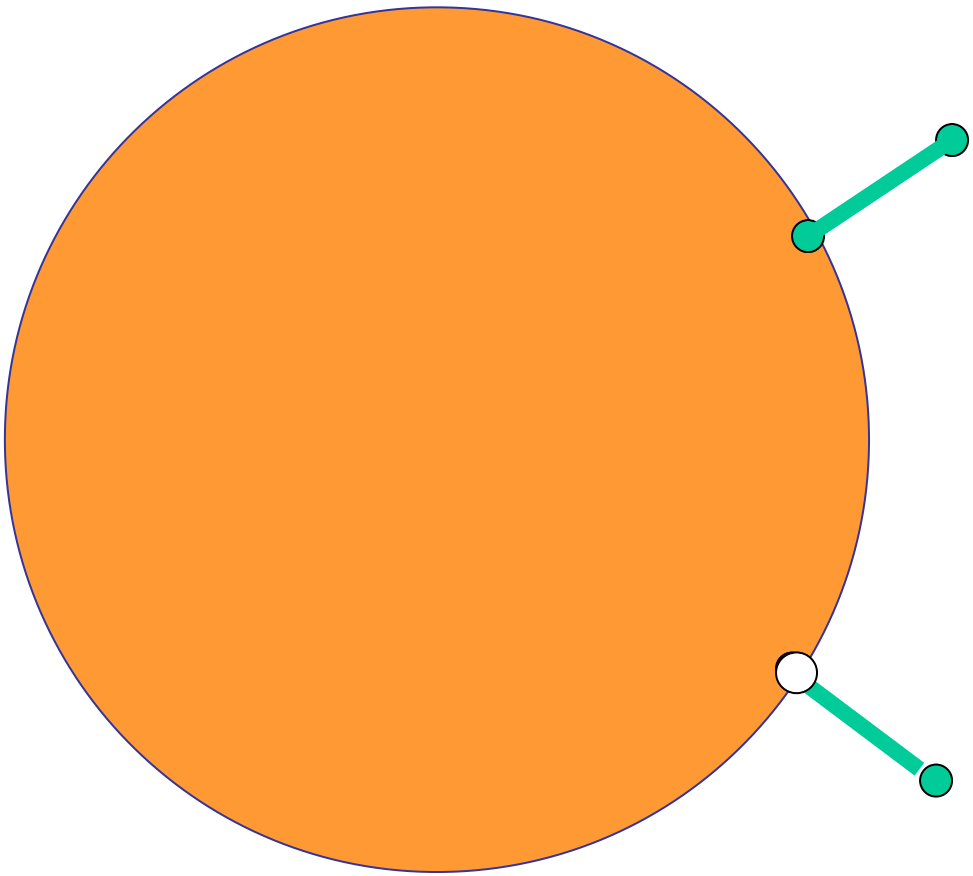
- No chiral breaking
- No confinement
- No generation of masses

→ Trivial theory ?

Grand potential unchanged: $(F = E - \mu N)$

- Adding a particle to the Fermi surface
- Taking out a particle (creating a hole)

$$F \rightarrow (E \pm E_F) - \mu(N \pm 1) = F$$



For an arbitrary attractive interaction it is convenient to form pairs particle-particle or hole-hole (Cooper pairs)

$$E + (\pm 2E_F - E_B) - \mu(N \pm 2) = F - E_B$$

In matter SC only under particular conditions (phonon interaction should overcome the Coulomb force)

$$\frac{T_c(\text{electr.})}{E(\text{electr.})} \approx \frac{1 \div 10\ {}^0K}{10^4 \div 10^5\ {}^0K} \approx 10^{-3} \div 10^{-4}$$

In QCD attractive interaction (antitriplet channel) $\frac{T_c(\text{quarks})}{E(\text{quarks})} \approx \frac{50\ \text{MeV}}{100\ \text{MeV}} \approx 1$

SC much more efficient in QCD

Effective theory

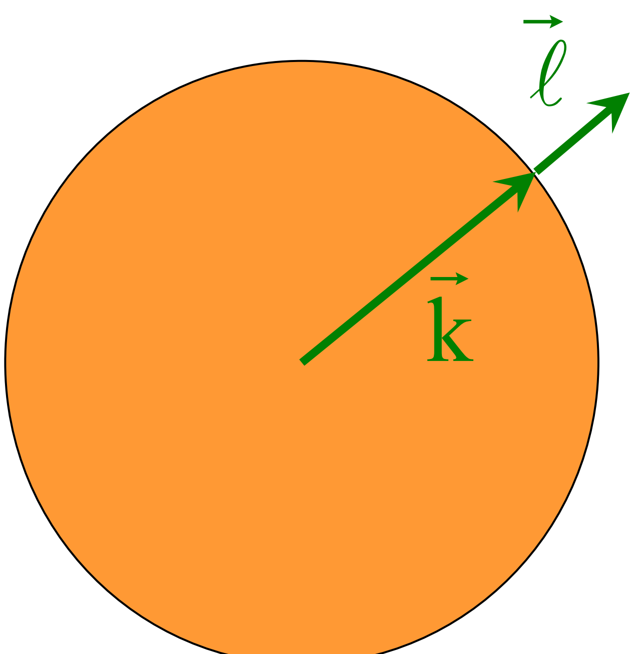
- Field theory at the Fermi surface
- The free fermion gas
- One-loop corrections

Field theory at the Fermi surface

(Polchinski, TASI 1992, hep-th/9210046)

Renormalization group analysis a la Wilson

How do fields behave scaling down the energies toward ε_F by a factor $s<1$?



$$\vec{p} = \vec{k} + \vec{\ell}$$

Scaling:

$$E \Rightarrow sE$$
$$\vec{\ell} \Rightarrow s\vec{\ell}$$
$$\vec{k} \Rightarrow \vec{k}$$

Using the invariance under phase transformations, construction of the most general action for the effective degrees of freedom: particles and holes close to the Fermi surface (non-relativistic description)

$$\int dt\, d^3\vec{p}\left[i\psi_\sigma^\dagger(\vec{p})\,\partial_t\psi_\sigma(\vec{p}) - \left(\varepsilon(\vec{p})-\varepsilon_F\right)\psi_\sigma^\dagger(\vec{p})\,\psi_\sigma(\vec{p})\right]$$

Expanding around ε_F:

$$\varepsilon(\vec{p})-\varepsilon_F = \left|\frac{\partial\varepsilon(\vec{p})}{\partial\vec{p}}\right|_{\ell=0}\cdot\vec{\ell}+O\left(\ell^2\right)\equiv v_F\ell+\cdots$$

$$S=\int dt\,d^2\vec{k}\,d\vec{\ell}\left[i\psi_\sigma^\dagger(\vec{p})\partial_t\psi_\sigma(\vec{p})-\ell v_F\psi_\sigma^\dagger(\vec{p})\psi_\sigma(\vec{p})\right]$$

Scaling: $\Rightarrow$ $S\rightarrow s^{2d_\psi+1}S$

$$\ell\rightarrow s\ell$$

$$dt\rightarrow s^{-1}dt$$

$$d\vec{k}\rightarrow d\vec{k}$$

$$d\vec{\ell}\rightarrow s\,d\vec{\ell}$$

$$\partial_t\rightarrow s\,\partial_t$$

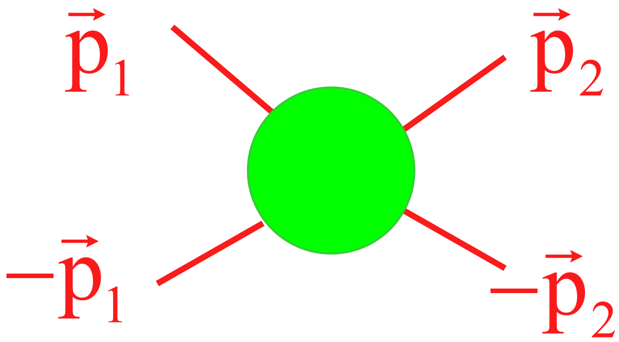
requiring the action S to be invariant

$$\psi\rightarrow s^{-1/2}\psi$$

The result of the analysis is that all possible interaction terms are irrelevant (go to zero going toward the Fermi surface) except a marginal (independent on s) quartic interaction of the form:

$$V\sum_{\sigma,\sigma'}\int dt\, d^3\vec{p}_1 d^3\vec{p}_2 \psi^\dagger_\sigma(\vec{p}_1)\psi_\sigma(\vec{p}_2)\psi^\dagger_{\sigma'}(-\vec{p}_1)\psi_{\sigma'}(-\vec{p}_2)$$

corresponding to a Cooper-like interaction



Quartic

s^{-1+4}

$$\int dt\, d^2\vec{k}_1 d\vec{\ell}_1 d^2\vec{k}_2 d\vec{\ell}_2 d^2\vec{k}_3 d\vec{\ell}_3 d^2\vec{k}_4 d\vec{\ell}_4$$

$$V(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4)\psi^+_\sigma(\vec{p}_1)\psi_\sigma(\vec{p}_3)\psi^+_{\sigma'}(\vec{p}_2)\psi_{\sigma'}(\vec{p}_4)$$

$$\delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4)$$

$s^{-4x1/2}$

s^{δ} ??

Scales as $s^{1+\delta}$

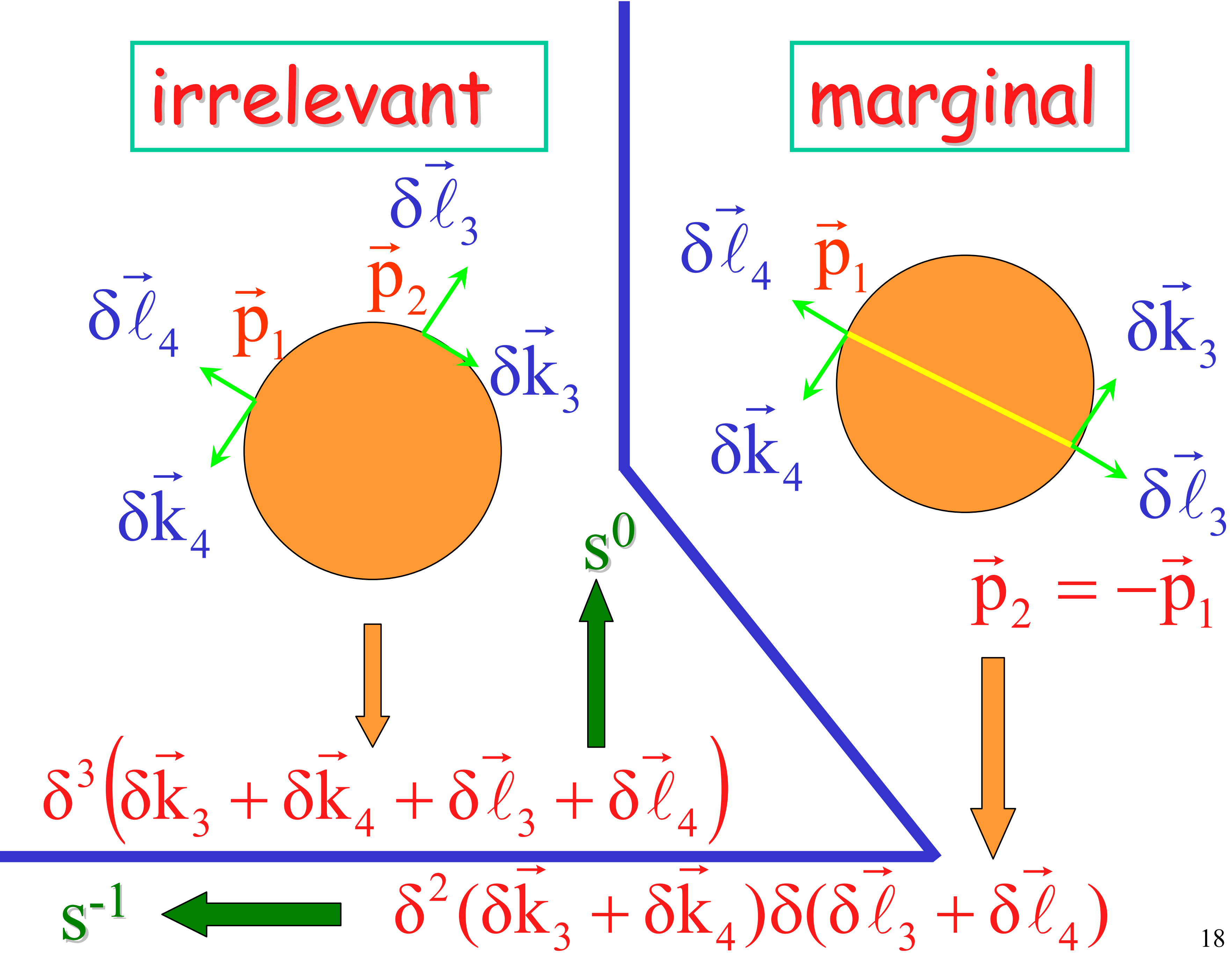
Scattering:

$$\vec{p}_1 + \vec{p}_2 \rightarrow \vec{p}_3 + \vec{p}_4$$

$$\vec{p}_3 = \vec{p}_1 + \delta\vec{k}_3 + \delta\vec{\ell}_3$$

$$\vec{p}_4 = \vec{p}_2 + \delta\vec{k}_4 + \delta\vec{\ell}_4$$

$$\delta^3\left(\delta\vec{k}_3 + \delta\vec{k}_4 + \delta\vec{\ell}_3 + \delta\vec{\ell}_4\right)$$


irrelevant
marginal
$\delta\vec{\ell}_3$
$\vec{p}_2$
$\delta\vec{\ell}_4$
$\vec{p}_1$
$\delta\vec{k}_3$
$\delta\vec{k}_4$
s^0
$\delta^3\left(\delta\vec{k}_3+\delta\vec{k}_4+\delta\vec{\ell}_3+\delta\vec{\ell}_4\right)$
$\delta\vec{\ell}_4$
$\vec{p}_1$
$\delta\vec{k}_3$
$\delta\vec{k}_4$
$\delta\vec{\ell}_3$
$\vec{p}_2=-\vec{p}_1$
s^{-1}
$\delta^2(\delta\vec{k}_3+\delta\vec{k}_4)\delta(\delta\vec{\ell}_3+\delta\vec{\ell}_4)$

Higher order interactions irrelevant

Free theory BUT check quantum corrections to the marginal interactions among the Cooper pairs

The free fermion gas

Eq. of motion: $$(i\partial_t - \ell v_F)\psi_\sigma(\vec{p},t) = 0$$

Propagator: $$(i\partial_t - \ell v_F)G_{\sigma,\sigma'}(\vec{p},t) = \delta_{\sigma,\sigma'}\delta(t)$$

$$G_{\sigma,\sigma'}(\vec{p},t) = \delta_{\sigma,\sigma'}G(\vec{p},t) =$$
$$= -i\delta_{\sigma,\sigma'}\left[\theta(t)\theta(\ell) - \theta(-t)\theta(-\ell)\right]e^{-i\ell v_F t}$$

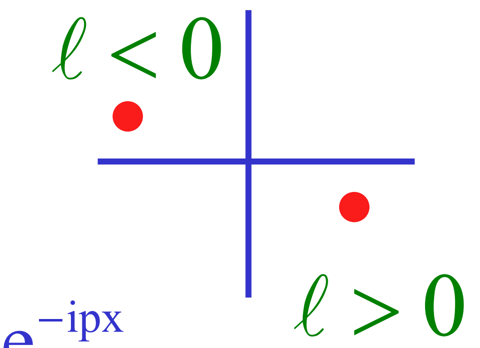
Using: $$\theta(t) = \frac{i}{2\pi}\int d\omega \frac{e^{-i\omega t}}{\omega + i\varepsilon}$$

$$G(\vec{p},t)=\lim_{\varepsilon\to 0^{+}}\frac{1}{2\pi}\int d\omega\, e^{-i\omega t}\left[\frac{\theta(\ell)}{\omega-\ell v_F+i\varepsilon}+\frac{\theta(-\ell)}{\omega-\ell v_F-i\varepsilon}\right]$$

or:

$$G(\vec{p},t)\equiv\frac{1}{2\pi}\int dp_0\, e^{-ip_0 t}G(p_0,\vec{p})$$

$$G(p)=\frac{1}{(1+i\varepsilon)p_0-\ell v_F}$$



Fermi field decomposition

$$\psi_\sigma(x)=\sum_{\vec{p}}b_\sigma(\vec{p},t)e^{i\vec{p}\cdot\vec{x}}=\sum_{\vec{p}}b_\sigma(\vec{p})e^{-ipx}$$

$$x^\mu=(t,\vec{x}),\quad p^\mu=(\ell v_F,\vec{p})$$

with:

$$b_\sigma(\vec{p})\left|0\right\rangle = 0 \quad \text{for} \quad |\vec{p}| > p_F$$

$$b_\sigma^\dagger(\vec{p})\left|0\right\rangle = 0 \quad \text{for} \quad |\vec{p}| < p_F$$

$$[b_\sigma(\vec{p}), b_\sigma^\dagger(\vec{p})]_+ = \delta_{\vec{p},\vec{p}'}\delta_{\sigma,\sigma'}$$

$$[\psi_\sigma(\vec{x},t), \psi_\sigma^\dagger(\vec{x}',t)]_+ = \delta_{\sigma,\sigma'}\delta^3(\vec{x}-\vec{x}')$$

The following representation holds:

$$G_{\sigma,\sigma'}(x) = -i\delta_{\sigma,\sigma'}\sum_{\vec{p}}\left\langle 0\right|T(b_\sigma(\vec{p},t)b_\sigma^\dagger(\vec{p},0)\left|0\right\rangle e^{i\vec{p}\cdot\vec{x}} = \delta_{\sigma,\sigma'}\sum_{\vec{p}} G(\vec{p},t)$$

In fact, using

$$\left\langle 0\right|b_\sigma^\dagger(\vec{p})b_\sigma(\vec{p})\left|0\right\rangle = \theta(p_F - p) = \theta(-\ell)$$

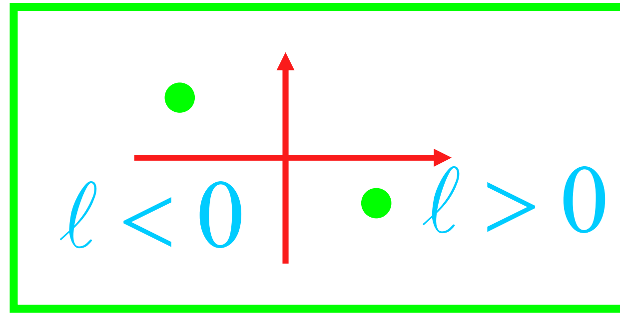
$$\left\langle 0\right|b_\sigma(\vec{p})b_\sigma^\dagger(\vec{p})\left|0\right\rangle = 1-\theta(p_F - p) = \theta(p - p_F) = \theta(\ell)$$

$$G(\vec{p},t) = \begin{cases} -i\theta(\ell)e^{-i\ell v_F t}, & t>0 \\ i\theta(-\ell)e^{-i\ell v_F t}, & t<0 \end{cases}$$

The following property is useful:

$$\lim_{\delta\to 0^+} G_{\sigma,\sigma}(\vec{0},-\delta) = -i\lim_{\delta\to 0^+}\left\langle 0\right| T(\psi_\sigma(\vec{0},-\delta)\psi^\dagger_\sigma(0)\left|0\right\rangle =$$

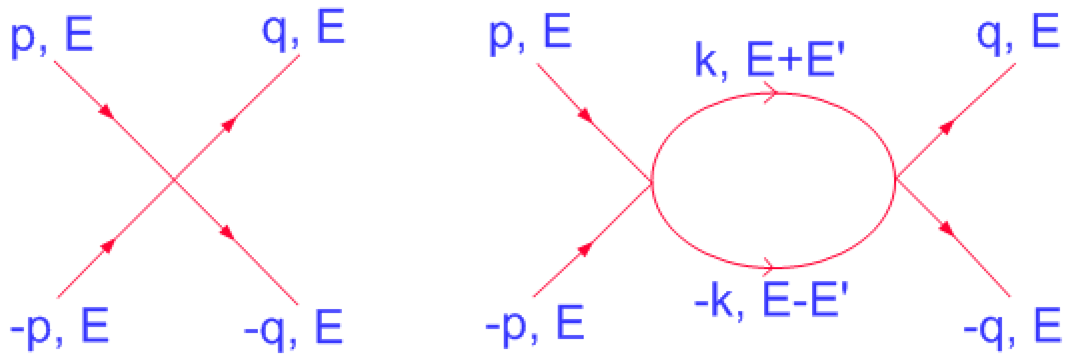
$$= i\left\langle 0\right|\psi^\dagger_\sigma\psi_\sigma\left|0\right\rangle \equiv i\rho_F$$



$$\rho_F = -2i\lim_{\delta\to 0^+}\sum_\sigma G_{\sigma,\sigma}(\vec{0},-\delta) = -2i\lim_{\delta\to 0^+}\int\frac{d^4p}{(2\pi)^4}e^{ip_0\delta}\frac{1}{(1+i\varepsilon)p_0-\ell v_F}$$

$$\rho_F = 2\int\frac{d^3\vec{p}}{(2\pi)^3}\theta(-\ell) = 2\int\frac{d^3\vec{p}}{(2\pi)^3}\theta(p_F-p) = \frac{p_F^3}{3\pi^2}$$

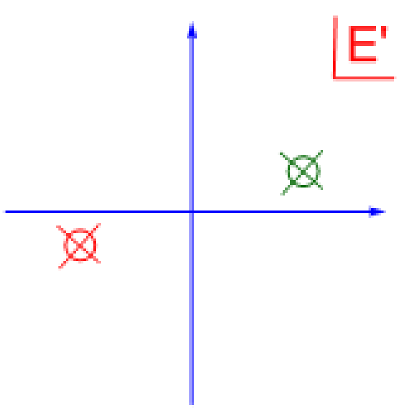
One-loop corrections



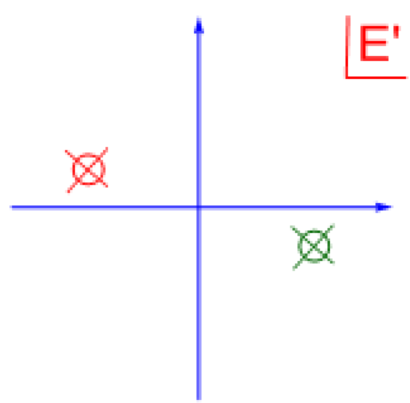
$$\frac{1}{(1+i\varepsilon)p_0 - \ell v_F}$$

$$iG(E) = iG - G^2 \int \frac{dE' d^2\vec{k} d\ell}{(2\pi)^4} \frac{1}{((E+E')(1+i\varepsilon) - v_F \ell)((E-E')(1+i\varepsilon) - v_F \ell)}$$

l > 0



l < 0



Closing in the upper plane we get

$$G(E) = G - \frac{1}{2} G^2 \rho \log(\delta/E) + O(G^3)$$

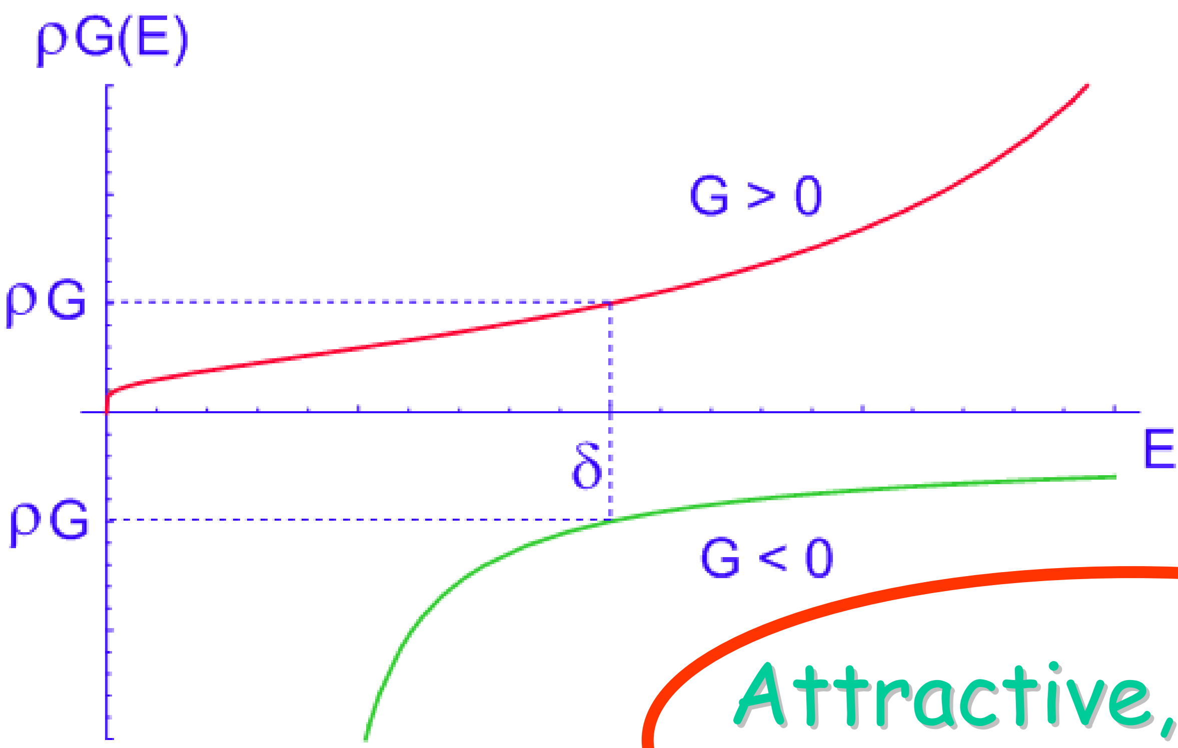
$$\rho = 2\int \frac{d^2\vec{k}}{(2\pi)^3} \frac{1}{v_F(\vec{k})}$$

δ, UV cutoff on $v_F \ell$

From RG equations:

$$\frac{dG(E)}{dE} = \frac{1}{2E} \rho G(E)^2$$

$$\rho G(E) = \frac{\rho G}{1 + \frac{\rho G}{2}\log(\delta/E)}$$



$E \to 0$

BCS instability

Attractive, stronger for $E \to 0$

Functional approach

$$S\left[\psi,\psi^\dagger\right]=\int d^4x\left[\psi^\dagger(i\partial_t-\varepsilon(|\vec{\nabla}|)+\mu)\psi+\frac{G}{2}(\psi^\dagger\psi)^2\right]$$

Fierzing $(C=i\sigma_2)$

$$\psi_a^\dagger\psi_a\psi_b^\dagger\psi_b=-\psi_a^\dagger\psi_b^\dagger\psi_a\psi_b=$$

$$=-\frac{1}{4}\varepsilon_{ab}\varepsilon_{ab}\psi_c^\dagger\psi^{\dagger c}\psi_d\psi^d=-\frac{1}{2}\psi^\dagger C\psi^*\psi^T C\psi$$

$$S\left[\psi,\psi^\dagger\right]=\int d^4x\left[\psi^\dagger(i\partial_t-\varepsilon(|\vec{\nabla}|)+\mu)\psi-\frac{G}{4}(\psi^\dagger C\psi^*)(\psi^T C\psi)\right]$$

Quantum theory

$$Z=\int D(\psi,\psi^\dagger)e^{iS\left[\psi,\psi^\dagger\right]}$$

$$\text{const.} = \int D(\Delta,\Delta^*) e^{-\frac{i}{G}\int d^4x \left[\Delta - \frac{G}{2}(\psi^T C \psi)\right]\left[\Delta^* + \frac{G}{2}(\psi^\dagger C \psi^*)\right]}$$

$$\frac{Z}{Z_0} = \frac{1}{Z_0}\int D(\psi,\psi^\dagger) D(\Delta,\Delta^*) e^{iS_0[\psi,\psi^\dagger] + i\int d^4x \left[-\frac{|\Delta|^2}{G} - \frac{1}{2}\Delta(\psi^\dagger C \psi^*) + \frac{1}{2}\Delta^*(\psi^T C \psi)\right]}$$

$$\chi = \frac{1}{\sqrt{2}}\begin{pmatrix} \psi \\ C\psi^* \end{pmatrix}$$

$$S_0 + \cdots = \int d^4x \left(\chi^\dagger S^{-1} \chi - \frac{|\Delta|^2}{G} \right)$$

$$S^{-1}(p) = \begin{bmatrix} p_0 - \xi_p & -\Delta \\ -\Delta^* & p_0 + \xi_p \end{bmatrix}$$

Since ψ^* appears already in χ we are double-counting. Solution: integrate over the fermions with the "replica trick":

$$\frac{Z}{Z_0} = \frac{1}{Z_0}\int D(\Delta,\Delta^*)\left[\det(S^{-1})\right]^{1/2} e^{-i\int d^4x \frac{|\Delta|^2}{G}} \equiv e^{iS_{eff}}$$

$$S_{eff}(\Delta,\Delta^*) = -\frac{i}{2}\mathrm{Tr}\left[\log(S_0 S^{-1})\right] - \int d^4x \frac{|\Delta|^2}{G}$$

Evaluating the saddle point:

$$\Delta = iG\int \frac{d^4p}{(2\pi)^4}\frac{\Delta}{p_0^2 - \xi_p^2 - |\Delta|^2} \Longrightarrow \Delta = \frac{G}{2}\int \frac{d^3p}{(2\pi)^3}\frac{\Delta}{\sqrt{\xi_p^2 + |\Delta|^2}}$$

At T not 0, introducing the Matsubara frequencies

$$\omega_n = (2n+1)\pi T$$

$$\Delta = GT\sum_{n=-\infty}^{+\infty}\int\frac{d^3p}{(2\pi)^3}\frac{\Delta}{\omega_n^2+\xi_p^2+|\Delta|^2}$$

and using (f(E) is the Fermi particle density)

$$\sum_{n=-\infty}^{+\infty}\frac{1}{\omega_n^2+\xi_p^2+|\Delta|^2} = \frac{1}{2E_pT}\underbrace{(1-2f(E_p))}$$

$$\boxed{\Delta = \frac{G}{2}\int\frac{d^3p}{(2\pi)^3}\frac{\Delta}{\sqrt{\xi_p^2+|\Delta|^2}}\tanh(E_p/2T)}$$

By saddle point:

$$\frac{Z}{Z_0} = \frac{1}{Z_0}\int D(\psi,\psi^\dagger)D(\Delta,\Delta^*)e^{iS_o[\psi,\psi^\dagger]+i\int d^4x\left[-\frac{|\Delta|^2}{G}-\frac{1}{2}\Delta(\psi^\dagger C\psi^*)+\frac{1}{2}\Delta^*(\psi^T C\psi)\right]}$$

$$\Delta = \frac{G}{2}\left\langle \psi^T C\psi \right\rangle$$

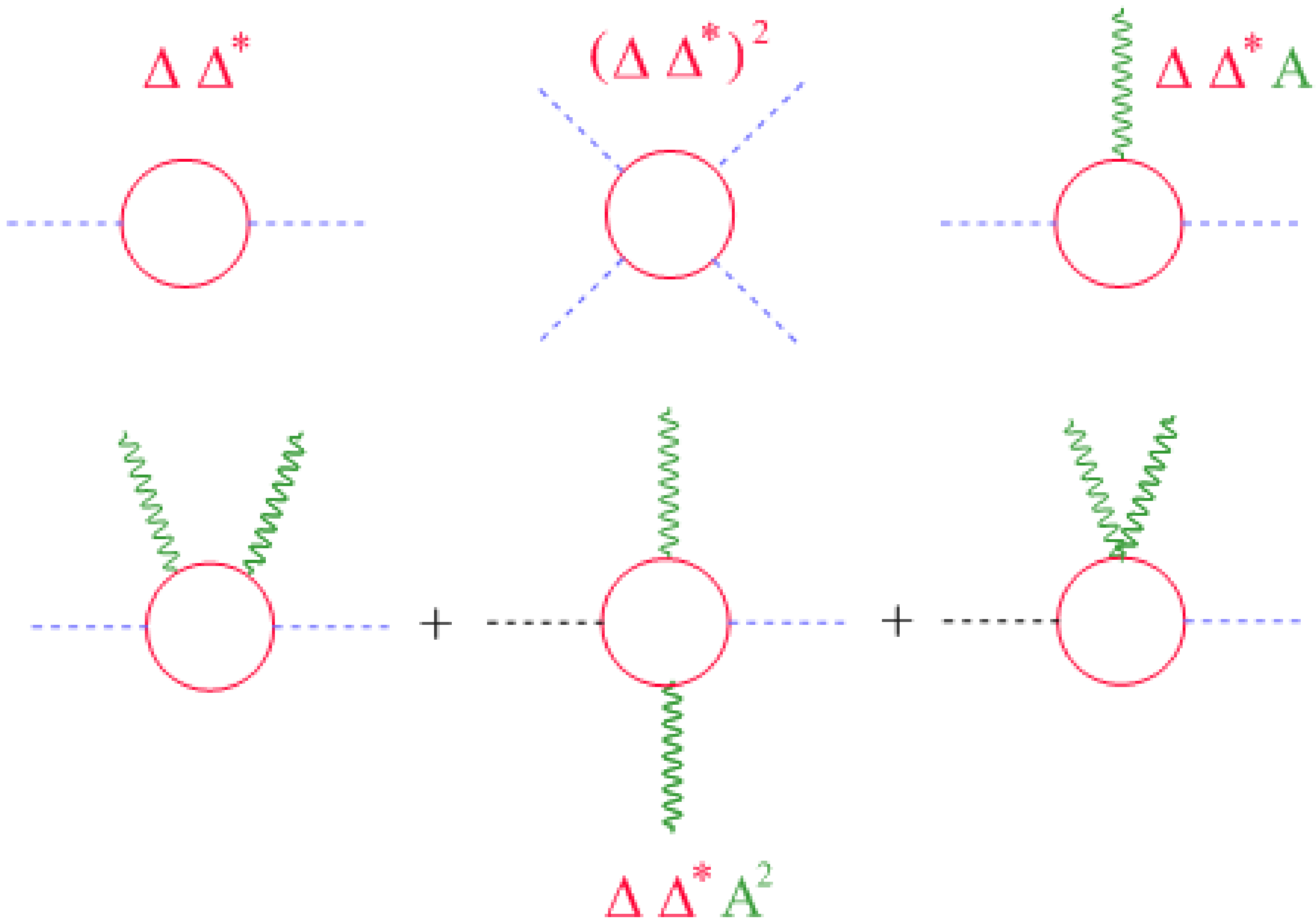
Introducing the em interaction in S_0 we see that Z is gauge invariant under

$$\psi \to \psi e^{i\alpha(x)}, \quad \Delta \to \Delta e^{2i\alpha(x)}$$

Therefore also S_{eff} must be gauge invariant and it will depend on the space-time derivatives of Δ through

$$D_\mu = \partial_\mu + 2ieA_\mu$$

In fact, evaluating the diagrams (Gor'kov 1959):



got the result (with a convenient renormalization of the fields):

$$H = \int d^3\vec{r} \left(-\frac{1}{4m} \psi^*(\vec{r}) \,|\, (\vec{\nabla} + i2e\vec{A}) \,|^2\, \psi(\vec{r}) + \alpha \,|\, \psi(\vec{r}) \,|^2 + \frac{1}{2} \beta \,|\, \psi(\vec{r}) \,|^4 \right)$$

charge of the pair

This result gave full justification to the Landau treatment of superconductivity

The critical temperature

By definition at T_c the gap vanishes. One can perform a GL expansion of the grand potential

$$\Omega = \frac{1}{2}\alpha\Delta^2 + \frac{1}{4}\beta\Delta^4$$

with extrema: $\alpha\Delta + \beta\Delta^3 = 0$

α and β from the expansion of the gap equation up to normalization

$$\Delta = GT\sum_{n=-\infty}^{+\infty}\int\frac{d^3p}{(2\pi)^3}\frac{\Delta}{\omega_n^2 + \xi_p^2 + |\Delta|^2}$$

To get the normalization remember (in the weak coupling and relatively to the normal state):

$$\langle H_0 \rangle = \Omega = -\frac{1}{4}\rho\Delta^2$$

Starting from the gap equation: $\Delta - \frac{1}{2}\rho G\Delta \log\frac{2\delta}{\Delta} = 0$

Integrating over Δ and using the gap equation one finds:

$$-\frac{G\rho}{8}\Delta^2$$

Rule to get the effective potential from the gap equation: Integrate the gap equation over Δ and multiply by 2/G

Expanding the gap equation in Δ: $(\omega_n = (2n+1)\pi T)$

$$\Delta - 2G\rho T\,\mathrm{Re}\sum_{n=0}^{\infty}\int_0^{\delta} d\xi\left[\frac{\Delta}{(\omega_n^2+\xi^2)} - \frac{\Delta^3}{(\omega_n^2+\xi^2)^2} + \cdots\right] = 0$$

One gets:

$$\alpha = \frac{2}{G}\left(1 - 2G\rho T\,\mathrm{Re}\sum_{n=0}^{\infty}\int_0^{\delta}\frac{d\xi}{(\omega_n^2+\xi^2)}\right)$$

Integrating over ξ and summing over n up to N

$$\beta = 4\rho T\,\mathrm{Re}\sum_{n=0}^{\infty}\int_0^{\delta}\frac{d\xi}{(\omega_n^2+\xi^2)^2}$$

$$\omega_N = \delta \Rightarrow N \approx \frac{\delta}{2\pi T}$$

$$\alpha(T) = \rho \log \frac{\pi T}{\gamma \Delta_0} \qquad \gamma = e^C, \quad C = 0.577...$$

Requiring $\alpha(T_c) = 0$

$$T_c = \frac{\gamma}{\pi}\Delta_0 \approx 0.56693\,\Delta_0$$

Also

$$\beta(T) = \frac{7\rho}{8\pi^2 T^2}\zeta(3)$$

and, from the gap equation

$$\left(\alpha(T) \approx -\rho\left(1 - \frac{T}{T_c}\right)\right)$$

$$\Delta^2(T) = -\frac{\alpha(T)}{\beta(T)} \Rightarrow \Delta(T) \approx \frac{2\sqrt{2}\pi T_c}{\sqrt{7\zeta(3)}}\left(1 - \frac{T}{T_c}\right)^{1/2} \approx 3.06 T_c \left(1 - \frac{T}{T_c}\right)^{1/2}$$