Università degli Studi di Perugia - Corso di Laurea Triennale in Fisica

Corso di

MECCANICA QUANTISTICA

Oscillatore armonico

$$a=\frac{1}{\sqrt{2m\hbar\omega}}\left(ip+m\omega x\right)\ ,\qquad a^{\dagger}=\frac{1}{\sqrt{2m\hbar\omega}}\left(-ip+m\omega x\right)\ ,$$

$$H=\left(a^{\dagger}a+\frac{1}{2}\right)\hbar\omega=\left(aa^{\dagger}-\frac{1}{2}\right)\hbar\omega\ ,\qquad\left[a,a^{\dagger}\right]=1\ ,$$

$$[H,a]=-\hbar\omega a\ ,\qquad\left[H,a^{\dagger}\right]=\hbar\omega a^{\dagger}\ .$$

Se $H\psi=E\psi$, allora

$$Ha\psi=(E-\hbar\omega)\,a\psi\ ,\qquad Ha^{\dagger}\psi=(E+\hbar\omega)\,a^{\dagger}\psi\ ,$$

inoltre

$$0\leq\left(\psi,a^{\dagger}a\psi\right)=\left(\psi,\left(\frac{H}{\hbar\omega}-\frac{1}{2}\right)\psi\right)=\left(\frac{E}{\hbar\omega}-\frac{1}{2}\right)(\psi,\psi)\ ,$$

quindi

$$E\geq\frac{\hbar\omega}{2}\ ,\qquad\left(E=\frac{\hbar\omega}{2}\ \text{ se }\ a\psi=0\right)\ .$$

$$H\psi_n=E_n\psi_n\ ,\qquad E_n=\left(n+\frac{1}{2}\right)\hbar\omega\quad(n=0,1,2,\dots)\ .$$

Se gli autostati ψ_n sono normalizzati si può scrivere:

$$a^{\dagger}\psi_n=\sqrt{n+1}\,\psi_{n+1}\ ,\qquad a\psi_n=\sqrt{n}\,\psi_{n-1}\ ,\qquad(a\psi_0=0)\ ,$$

da cui segue

$$\psi_n=\frac{1}{\sqrt{n!}}\left(a^{\dagger}\right)^n\psi_0\ .$$

Per trovare l'espressione esplicita degli ψ_n si noti che

$$a=\frac{1}{\sqrt{2}}\left(\frac{d}{d\xi}+\xi\right)=\frac{1}{\sqrt{2}}e^{-\frac{\xi^2}{2}}\frac{d}{d\xi}e^{\frac{\xi^2}{2}}\ ,\qquad a^{\dagger}=\frac{1}{\sqrt{2}}\left(-\frac{d}{d\xi}+\xi\right)=-\frac{1}{\sqrt{2}}e^{\frac{\xi^2}{2}}\frac{d}{d\xi}e^{-\frac{\xi^2}{2}}\ ,$$

dove

$$\xi=\alpha x\ ,\qquad\alpha=\sqrt{\frac{m\omega}{\hbar}}\ .$$

Da $a\psi_0=0$ segue: $\frac{d}{d\xi}\left(e^{\frac{\xi^2}{2}}\psi_0\right)=0$, quindi $\psi_0=\sqrt{\frac{\alpha}{\sqrt{\pi}}}e^{-\frac{\xi^2}{2}}$. Per le autofunzioni ($n=0,1,2,\dots$) abbiamo quindi:

$$\psi_n=\frac{1}{\sqrt{n!}}\left(a^{\dagger}\right)^n\psi_0=\sqrt{\frac{\alpha}{2^n n!\sqrt{\pi}}}e^{-\frac{\xi^2}{2}}\cdot(-1)^n e^{\frac{\xi^2}{2}}\frac{d^n}{d\xi^n}e^{-\frac{\xi^2}{2}}=N_nH_n(\xi)e^{-\frac{\xi^2}{2}}$$

Rappresentazione matriciale dei vari operatori nella base delle ψ_n:

$$(a)_{nm} \equiv (\psi_n, a\psi_m) = \sqrt{n+1}\,\delta_{m,n+1} \ , \qquad (a^\dagger)_{nm} \equiv (\psi_n, a^\dagger\psi_m) = \sqrt{n}\,\delta_{m,n-1}$$

$$(a)_{nm} = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & 0 & \dots \\ 0 & 0 & \sqrt{2} & 0 & 0 & \dots \\ 0 & 0 & 0 & \sqrt{3} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \ , \qquad (a^\dagger)_{nm} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ \sqrt{1} & 0 & 0 & 0 & 0 & \dots \\ 0 & \sqrt{2} & 0 & 0 & 0 & \dots \\ 0 & 0 & \sqrt{3} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$$x = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right) \ , \qquad p = -i\sqrt{\frac{m\hbar\omega}{2}}\left(a - a^\dagger\right) \ ,$$

$$(x)_{nm} = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n+1}\,\delta_{m,n+1} + \sqrt{n}\,\delta_{m,n-1}\right) \ ,$$

$$(p)_{nm} = i\sqrt{\frac{m\hbar\omega}{2}}\left(-\sqrt{n+1}\,\delta_{m,n+1} + \sqrt{n}\,\delta_{m,n-1}\right) \ .$$

Tenendo conto che, per esempio $(x^2)_{nm} = \sum_k (x)_{nk}(x)_{km}$, si ottiene:

$$(x^2)_{nm} = \frac{\hbar}{2m\omega}\left(\sqrt{(n+1)(n+2)}\,\delta_{m,n+2} + (2n+1)\,\delta_{m,n} + \sqrt{n(n-1)}\,\delta_{m,n-2}\right) \ ,$$

$$(p^2)_{nm} = \frac{m\hbar\omega}{2}\left(-\sqrt{(n+1)(n+2)}\,\delta_{m,n+2} + (2n+1)\,\delta_{m,n} - \sqrt{n(n-1)}\,\delta_{m,n-2}\right) \ .$$